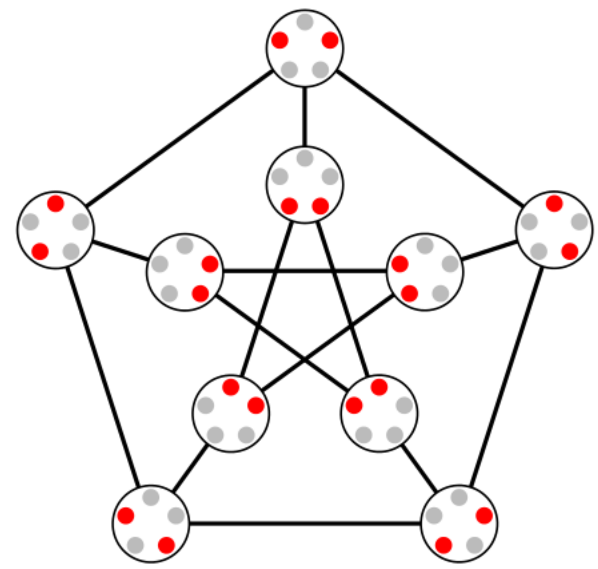
Kneser graph

In graph theory, the **Kneser graph** $KG_{n,k}$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Kneser graphs are named after Martin Kneser, who first investigated them in 1955.

Kneser graph



The Kneser graph $KG_{5,2}$, isomorphic to the Petersen graph

Named after	Martin Kneser
Vertices	$\binom{n}{k}$
Edges	$\binom{n}{k}\binom{n-k}{k}/2$
Chromatic number	$\begin{cases} n-2k+2 & n \geq 2k \\ 1 & \text{otherwise} \end{cases}$
Properties	$\binom{n-k}{k}$-regular arc-transitive
Notation	$KG_{n,k}$, $K(n,k)$

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The complete graph on n vertices is the Kneser graph $KG_{n,1}$.

The complement of the line graph of the complete graph on n vertices is the Kneser graph $KG_{n,2}$.

The Kneser graph $KG_{2n-1,n-1}$ is known as the odd graph O_n; the odd graph $O_3 = KG_{5,2}$ is isomorphic to the Petersen graph.

Properties

- The Kneser graph is vertex transitive and edge transitive. Each vertex has exactly $\binom{n-k}{k}$ neighbors. However, the Kneser graph is not, in general, a strongly regular graph, as different pairs of nonadjacent vertices have different numbers of common neighbors depending on the size of the intersection of the corresponding pair of sets.
- As Kneser (1955) conjectured, the chromatic number of the Kneser graph $KG_{n,k}$ is exactly $n - 2k + 2$; for instance, the Petersen graph requires three colors in any proper coloring. László Lovász (1978) proved this using topological methods, giving rise to the field of topological combinatorics. Soon thereafter Imre Bárány (1978) gave a simple proof, using the Borsuk–Ulam theorem and a lemma of David Gale, and Joshua E. Greene (2002) won the Morgan Prize for a further simplified but still topological proof. Jiří Matoušek (2004) found a purely combinatorial proof.
- When $n \geq (3k+1+\sqrt{5k^2-2k+1})/2$, the Kneser graph $KG_{n,k}$ always contains a Hamiltonian cycle (Chen 2003). Note that $(3k+1+\sqrt{5k^2-2k+1})/2 < 2.62k+1$, so this condition is satisfied if $n \geq 2.62k+1$. It is also known that all Kneser graphs $KG_{n,k}$ with $n = 2k + 2^a$ for any integers $k \geq 1$ and $a \geq 0$ have a Hamiltonian cycle (Mütze, Nummenpalo & Walczak 2018). In particular, the odd graph O_n has a Hamiltonian cycle for all $n \geq 4$. Moreover, computational searches have found that all connected Kneser graphs for $n \leq 27$, except for the Petersen graph, are Hamiltonian (Shields 2004).
- When $n < 3k$, the Kneser graph contains no triangles. More generally, when $n < ck$ it does not contain cliques of size c, whereas it does contain such cliques when $n \geq ck$. Moreover, although the Kneser graph always contains

cycles of length four whenever $n \geq 2k+2$, for values of n close to $2k$ the shortest odd cycle may have nonconstant length (Denley 1997).

- The diameter of a connected Kneser graph $KG_{n,k}$ is

$$\left\lceil \frac{k-1}{n-2k} \right\rceil + 1$$ (Valencia-Pabon & Vera 2005).

- The graph spectrum of the Kneser graph $KG_{n,k}$ is given as follows:

For $j = 0, \ldots, k$, the eigenvalue $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$ occurs with multiplicity $\binom{n}{j} - \binom{n}{j-1}$ for $j > 0$ and 1 for $j = 0$. See this (https://web.archive.org/web/20120323231232/http://www.math.caltech.edu/~2011-12/2term/ma192b/kneser-evals.pdf) paper for a proof.

- The Erdős–Ko–Rado theorem states that the independence number of the Kneser graph $KG_{n,k}$ is

$$\alpha(KG_{n,k}) = \binom{n-1}{k-1}$$

Related graphs

The Johnson graph $J_{n,k}$ is the graph whose vertices are the k-element subsets of an n-element set, two vertices being adjacent when they meet in a $(k-1)$-element set. For $k = 2$ the Johnson graph is the complement of the Kneser graph $KG_{n,2}$. Johnson graphs are closely related to the Johnson scheme, both of which are named after Selmer M. Johnson.

The **generalized Kneser graph** $KG_{n,k,s}$ has the same vertex set as the Kneser graph $KG_{n,k}$, but connects two vertices whenever they correspond to sets that intersect in s or fewer items (Denley 1997). Thus $KG_{n,k,0} = KG_{n,k}$.

The **bipartite Kneser graph** $H_{n,k}$ has as vertices the sets of k and $n-k$ items drawn from a collection of n elements. Two vertices are connected by an edge whenever one set is a subset of the other. Like the Kneser graph it is vertex transitive with degree $\binom{n-k}{k}$. The bipartite Kneser graph can be formed as a bipartite double cover of $KG_{n,k}$ in which one makes two copies of each vertex and replaces each edge by a pair of edges connecting corresponding pairs of vertices (Simpson 1991). The bipartite Kneser graph $H_{5,2}$ is the Desargues graph and the bipartite Kneser graph $H_{n,1}$ is a crown graph.

References

- Bárány, Imre (1978), "A short proof of Kneser's conjecture", *Journal of Combinatorial Theory*, Series A, **25** (3): 325–326, doi:10.1016/0097-3165(78)90023-7 (https://doi.org/10.1016%2F0097-3165%2878%2990023-7), MR 0514626 (https://www.ams.org/mathscinet-getitem?mr=0514626).
- Chen, Ya-Chen (2003), "Triangle-free Hamiltonian Kneser graphs", *Journal of Combinatorial Theory*, Series B, **89** (1): 1–16, doi:10.1016/S0095-8956(03)00040-6 (https://doi.org/10.1016%2FS0095-8956%2803%2900040-6), MR 1999733 (https://www.ams.org/mathscinet-getitem?mr=1999733).
- Denley, Tristan (1997), "The odd girth of the generalised Kneser graph", *European Journal of Combinatorics*, **18** (6): 607–611, doi:10.1006/eujc.1996.0122 (https://doi.org/10.1006%2Feujc.1996.0122), MR 1468332 (https://www.ams.org/mathscinet-getitem?mr=1468332).
- Greene, Joshua E. (2002), "A new short proof of Kneser's conjecture", *American Mathematical Monthly*, **109** (10): 918–920, doi:10.2307/3072460 (https://doi.org/10.2307%2F3072460), JSTOR 3072460 (https://www.jstor.org/sta

ble/3072460), MR 1941810 (https://www.ams.org/mathscinet-getitem?mr=1941810).

- Kneser, Martin (1955), "Aufgabe 360", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 2. Abteilung, **58**: 27.
- Lovász, László (1978), "Kneser's conjecture, chromatic number, and homotopy", *Journal of Combinatorial Theory*, Series A, **25** (3): 319–324, doi:10.1016/0097-3165(78)90022-5 (https://doi.org/10.1016%2F0097-3165%2878%2990022-5), MR 0514625 (https://www.ams.org/mathscinet-getitem?mr=0514625).
- Matoušek, Jiří (2004), "A combinatorial proof of Kneser's conjecture", *Combinatorica*, **24** (1): 163–170, doi:10.1007/s00493-004-0011-1 (https://doi.org/10.1007%2Fs00493-004-0011-1), MR 2057690 (https://www.ams.org/mathscinet-getitem?mr=2057690).
- Mütze, Torsten; Nummenpalo, Jerri; Walczak, Bartosz (2017), "Sparse Kneser graphs are Hamiltonian", arXiv:1711.01636 (https://arxiv.org/abs/1711.01636) [math.CO (https://arxiv.org/archive/math.CO)].
- Shields, Ian Beaumont (2004), *Hamilton Cycle Heuristics in Hard Graphs* (http://wayback.archive-it.org/all/20060917132442/http://www.lib.ncsu.edu/theses/available/etd-03142004-013420/), Ph.D. thesis, North Carolina State University, archived from the original (http://www.lib.ncsu.edu/theses/available/etd-03142004-013420/) on 2006-09-17.
- Simpson, J. E. (1991), "Hamiltonian bipartite graphs", *Proceedings of the Twenty-second Southeastern Conference on Combinatorics, Graph Theory, and Computing (Baton Rouge, LA, 1991)*, Congressus Numerantium, **85**, pp. 97–110, MR 1152123 (https://www.ams.org/mathscinet-getitem?mr=1152123).
- Valencia-Pabon, Mario; Vera, Juan-Carlos (2005), "On the diameter of Kneser graphs", *Discrete Mathematics*, **305** (1–3): 383–385, doi:10.1016/j.disc.2005.10.001 (https://doi.org/10.1016%2Fj.disc.2005.10.001), MR 2186709 (https://www.ams.org/mathscinet-getitem?mr=2186709).

External links

- Weisstein, Eric W. "Kneser Graph" (http://mathworld.wolfram.com/KneserGraph.html). *MathWorld*.
- Weisstein, Eric W. "Odd Graph" (http://mathworld.wolfram.com/OddGraph.html). *MathWorld*.