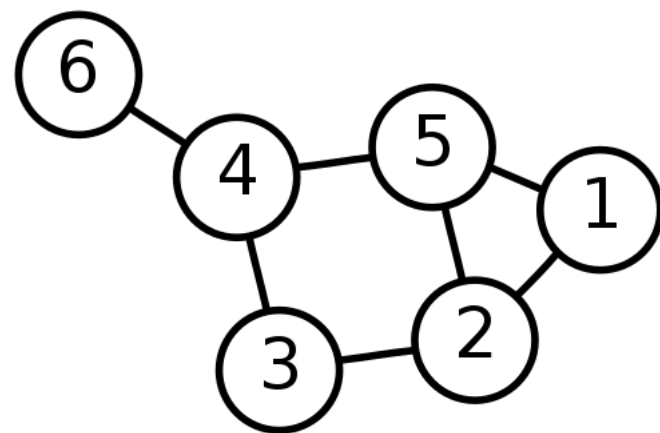


Graph (discrete mathematics)

In [mathematics](#), and more specifically in [graph theory](#), a **graph** is a structure amounting to a set of objects in which some pairs of the objects are in some sense "related". The objects correspond to mathematical abstractions called *vertices* (also called *nodes* or *points*) and each of the related pairs of vertices is called an *edge* (also called an *arc* or *line*).^[1] Typically, a graph is depicted in [diagrammatic form](#) as a set of dots for the vertices, joined by lines or curves for the edges. Graphs are one of the objects of study in [discrete mathematics](#).

The edges may be directed or undirected. For example, if the vertices represent people at a party, and there is an edge between two people if they shake hands, then this graph is undirected because any person *A* can shake hands with a person *B* only if *B* also shakes hands with *A*. In contrast, if any edge from a person *A* to a person *B* corresponds to *A*'s admiring *B*, then this graph is directed, because admiration is not necessarily reciprocated. The former type of graph is called an *undirected graph* and the edges are called *undirected edges* while the latter type of graph is called a *directed graph* and the edges are called *directed edges*.

Graphs are the basic subject studied by [graph theory](#). The word "graph" was first used in this sense by [James Joseph Sylvester](#) in 1878.^{[2][3]}



A drawing of a labeled graph on 6 vertices and 7 edges.

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Definitions

Definitions in graph theory vary. The following are some of the more basic ways of defining graphs and related mathematical structures.

Graph

In one very common sense of the term,^[4] a *graph* is an ordered pair $G = (V, E)$ comprising a set V of *vertices*, *nodes* or *points* together with a set E of *edges*, *arcs* or *lines*, which are 2-element subsets of V (i.e., an edge is associated with two vertices, and the association takes the form of the unordered pair of the vertices). To avoid ambiguity, this type of graph may be described precisely as undirected and simple.

Other senses of *graph* stem from different conceptions of the edge set. In one more general conception,^[5] E is a set together with a relation of *incidence* that associates with each edge two vertices. In another generalized notion, E is a multiset of unordered pairs of (not necessarily distinct) vertices. Many authors call these types of object multigraphs or pseudographs.

All of these variants and others are described more fully below.

The vertices belonging to an edge are called the *ends* or *end vertices* of the edge. A vertex may exist in a graph and not belong to an edge.

V and E are usually taken to be finite, and many of the well-known results are not true (or are rather different) for *infinite graphs* because many of the arguments fail in the infinite case. Moreover, V is often assumed to be non-empty, but E is allowed to be the empty set. The *order* of a graph is $|V|$, its number of vertices. The *size* of a graph is $|E|$, its number of edges. The *degree* or *valency* of a vertex is the number of edges that connect to it, where an edge that connects to the vertex at both ends (a loop) is counted twice.

For an edge $\{x, y\}$, graph theorists usually use the somewhat shorter notation xy .

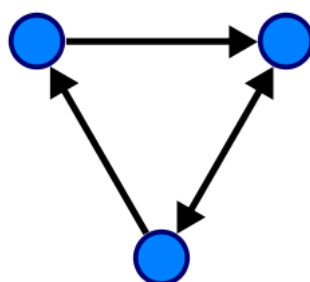
Adjacency relation

The edges E of an undirected graph G induce a symmetric binary relation \sim on V that is called the *adjacency relation* of G . Specifically, for each edge $\{x, y\}$, the vertices x and y are said to be *adjacent* to one another, which is denoted $x \sim y$.

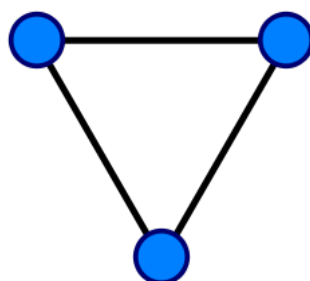
Types of graphs

Distinction in terms of the main definition

As stated above, in different contexts it may be useful to refine the term *graph* with different degrees of generality. Whenever it is necessary to draw a strict distinction, the following terms are used. Most commonly, in modern texts in graph theory, unless stated otherwise, *graph* means "undirected simple finite graph" (see the definitions below).



A directed graph.



A simple undirected graph with three vertices and three edges. Each vertex has degree two, so this is also a regular graph.

Undirected graph

An *undirected graph* is a graph in which edges have no orientation. The edge (x, y) is identical to the edge (y, x) . That is, they are not ordered pairs, but unordered pairs—i.e., sets of two vertices $\{x, y\}$ (or 2-multisets in the case of loops). The maximum number of edges in an undirected graph without a loop is $n(n - 1)/2$.

Directed graph

A *directed graph* or *digraph* is a graph in which edges have orientations. It is written as an ordered pair $G = (V, A)$ (sometimes $G = (V, E)$) with

- V a set whose elements are called *vertices*, *nodes*, or *points*;
- A a set of ordered pairs of vertices, called *arrows*, *directed edges* (sometimes simply *edges* with the corresponding set named E instead of A), *directed arcs*, or *directed lines*.

An arrow (x, y) is considered to be directed *from* x to y ; y is called the *head* and x is called the *tail* of the arrow; y is said to be a *direct successor* of x and x is said to be a *direct predecessor* of y . If a path leads from x to y , then y is said to be a *successor* of x and *reachable* from x , and x is said to be a *predecessor* of y . The arrow (y, x) is called the *inverted arrow* of (x, y) .

A directed graph G is called *symmetric* if, for every arrow in G , the corresponding inverted arrow also belongs to G . A symmetric loopless directed graph $G = (V, A)$ is equivalent to a simple undirected graph $G' = (V, E)$, where the pairs of inverse arrows in A correspond one-to-one with the edges in E ; thus the number of edges in G' is $|E| = |A|/2$, that is half the number of arrows in G .

Oriented graph

An *oriented graph* is a directed graph in which at most one of (x, y) and (y, x) may be arrows of the graph. That is, it is a directed graph that can be formed as an orientation of an undirected graph. However, some authors use "oriented graph" to mean the same as "directed graph".

Mixed graph

A *mixed graph* is a graph in which some edges may be directed and some may be undirected. It is written as an ordered triple $G = (V, E, A)$ with V , E , and A defined as above. Directed and undirected graphs are special cases.

Multigraph

Multiple edges are two or more edges that connect the same two vertices. A loop is an edge (directed or undirected) that connects a vertex to itself; it may be permitted or not, according to the application. In this context, an edge with two different ends is called a *link*.

A *multigraph*, as opposed to a simple graph, is an undirected graph in which multiple edges (and sometimes loops) are allowed.

Where graphs are defined so as to *disallow* both multiple edges and loops, a multigraph is often defined to mean a graph which can have both multiple edges and loops,^[6] although many use the term pseudograph for this meaning.^[7] Where graphs are defined so as to *allow* both multiple edges and loops, a multigraph is often defined to mean a graph without loops.^[8]

Simple graph

A simple graph is an undirected graph without multiple edges or loops. In a simple graph the edges form a *set* (rather than a multiset) and each edge is an unordered pair of *distinct* vertices. Thus, we can define a **simple graph** to be a set V of vertices together with a set E of edges, which are 2-element subsets of V .

In a simple graph with n vertices, the degree of every vertex is at most $n - 1$.

Quiver

A *quiver* or *multidigraph* is a directed multigraph. A quiver may have directed loops in it. Thus, a quiver is a set V of vertices, a set E of edges, and two functions $s : E \rightarrow V$, $t : E \rightarrow V$. The map s assigns to each edge its *source* (or *tail*), while the map t assigns to each edge its *target* (or *head*).

Weighted graph

A *weighted graph* is a graph in which a number (the weight) is assigned to each edge.^[9] Such weights might represent for example costs, lengths or capacities, depending on the problem at hand. Some authors call such a graph a *network*.^{[10][11]} Weighted correlation networks can be defined by soft-thresholding the pairwise correlations among

variables (e.g. gene measurements). Such graphs arise in many contexts, for example in shortest path problems such as the traveling salesman problem.

Half-edges, loose edges

In certain situations it can be helpful to allow edges with only one end, called *half-edges*, or no ends, called *loose edges*; see the articles Signed graphs and Biased graphs.

Important classes of graph

Regular graph

A *regular graph* is a graph in which each vertex has the same number of neighbours, i.e., every vertex has the same degree. A regular graph with vertices of degree k is called a k -regular graph or regular graph of degree k .

Complete graph

A *complete graph* is a graph in which each pair of vertices is joined by an edge. A complete graph contains all possible edges.

Finite graph

A *finite graph* is a graph in which the vertex set and the edge set are finite sets. Otherwise, it is called an *infinite graph*.

Most commonly in graph theory it is implied that the graphs discussed are finite. If the graphs are infinite, that is usually specifically stated.

Connected graph

In an undirected graph, an unordered pair of vertices $\{x, y\}$ is called *connected* if a path leads from x to y . Otherwise, the unordered pair is called *disconnected*.

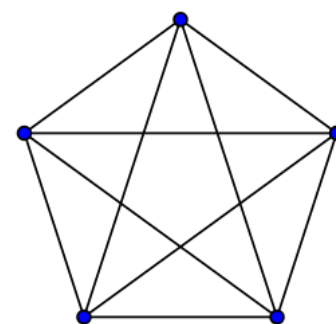
A *connected graph* is an undirected graph in which every unordered pair of vertices in the graph is connected. Otherwise, it is called a *disconnected graph*.

In a directed graph, an ordered pair of vertices (x, y) is called *strongly connected* if a directed path leads from x to y . Otherwise, the ordered pair is called *weakly connected* if an undirected path leads from x to y after replacing all of its directed edges with undirected edges. Otherwise, the ordered pair is called *disconnected*.

A *strongly connected graph* is a directed graph in which every ordered pair of vertices in the graph is strongly connected. Otherwise, it is called a *weakly connected graph* if every ordered pair of vertices in the graph is weakly connected. Otherwise it is called a *disconnected graph*.

A *k*-vertex-connected graph or *k*-edge-connected graph is a graph in which no set of $k - 1$ vertices (respectively, edges) exists that, when removed, disconnects the graph. A k -vertex-connected graph is often called simply a *k*-connected graph.

Bipartite graph



A complete graph with 5 vertices. Each vertex has an edge to every other vertex.

A bipartite graph is a graph in which the vertex set can be partitioned into two sets, W and X , so that no two vertices in W share a common edge and no two vertices in X share a common edge. Alternatively, it is a graph with a chromatic number of 2.

In a complete bipartite graph, the vertex set is the union of two disjoint sets, W and X , so that every vertex in W is adjacent to every vertex in X but there are no edges within W or X .

Path graph

A *path graph* or *linear graph* of order $n \geq 2$ is a graph in which the vertices can be listed in an order v_1, v_2, \dots, v_n such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, \dots, n - 1$. Path graphs can be characterized as connected graphs in which the degree of all but two vertices is 2 and the degree of the two remaining vertices is 1. If a path graph occurs as a subgraph of another graph, it is a path in that graph.

Planar graph

A *planar graph* is a graph whose vertices and edges can be drawn in a plane such that no two of the edges intersect.

Cycle graph

A *cycle graph* or *circular graph* of order $n \geq 3$ is a graph in which the vertices can be listed in an order v_1, v_2, \dots, v_n such that the edges are the $\{v_i, v_{i+1}\}$ where $i = 1, 2, \dots, n - 1$, plus the edge $\{v_n, v_1\}$. Cycle graphs can be characterized as connected graphs in which the degree of all vertices is 2. If a cycle graph occurs as a subgraph of another graph, it is a cycle or circuit in that graph.

Tree

A *tree* is a connected graph with no cycles.

A *forest* is a graph with no cycles, i.e. the disjoint union of one or more trees.

Advanced classes

More advanced kinds of graphs are:

- Petersen graph and its generalizations;
- perfect graphs;
- cographs;
- chordal graphs;
- other graphs with large automorphism groups: vertex-transitive, arc-transitive, and distance-transitive graphs;
- strongly regular graphs and their generalizations distance-regular graphs.

Properties of graphs

Two edges of a graph are called *adjacent* if they share a common vertex. Two arrows of a directed graph are called *consecutive* if the head of the first one is the tail of the second one. Similarly, two vertices are called *adjacent* if they share a common edge (*consecutive* if the first one is the tail and the second one is the head of an arrow), in which case the common edge is said to *join* the two vertices. An edge and a vertex on that edge are called *incident*.

The graph with only one vertex and no edges is called the *trivial graph*. A graph with only vertices and no edges is known as an *edgeless graph*. The graph with no vertices and no edges is sometimes called the *null graph* or *empty graph*, but the terminology is not consistent and not all mathematicians allow this object.

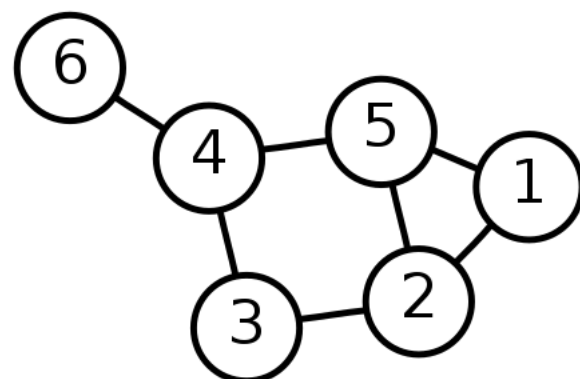
Normally, the vertices of a graph, by their nature as elements of a set, are distinguishable. This kind of graph may be called *vertex-labeled*. However, for many questions it is better to treat vertices as indistinguishable. (Of course, the vertices may be still distinguishable by the properties of the graph itself, e.g., by the numbers of incident edges.) The same remarks apply to edges, so graphs with labeled edges are called *edge-labeled*. Graphs with labels attached to edges or vertices are more generally designated as *labeled*. Consequently, graphs in which vertices are indistinguishable and edges are indistinguishable are called *unlabeled*. (Note that in the literature, the term *labeled* may apply to other kinds of labeling, besides that which serves only to distinguish different vertices or edges.)

The category of all graphs is the slice category $\text{Set} \downarrow D$ where $D: \text{Set} \rightarrow \text{Set}$ is the functor taking a set s to $s \times s$.

Examples

- The diagram at right is a graphic representation of the following graph:

$$V = \{1, 2, 3, 4, 5, 6\};$$



A graph with six nodes.

$$E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{4, 6\}\}.$$

- In category theory, a small category can be represented by a directed multigraph in which the objects of the category are represented as vertices and the morphisms as directed edges. Then, the functors between categories induce some, but not necessarily all, of the digraph morphisms of the graph.
- In computer science, directed graphs are used to represent knowledge (e.g., conceptual graph), finite state machines, and many other discrete structures.
- A binary relation R on a set X defines a directed graph. An element x of X is a direct predecessor of an element y of X if and only if xRy .
- A directed graph can model information networks such as Twitter, with one user following another.^{[12][13]}
- Particularly regular examples of directed graphs are given by the Cayley graphs of finitely-generated groups, as well as Schreier coset graphs

Graph operations

There are several operations that produce new graphs from initial ones, which might be classified into the following categories:

- *unary operations*, which create a new graph from an initial one, such as:
 - edge contraction,
 - line graph,
 - dual graph,
 - complement graph,
 - graph rewriting;

- binary operations, which create a new graph from two initial ones, such as:
 - disjoint union of graphs,
 - cartesian product of graphs,
 - tensor product of graphs,
 - strong product of graphs,
 - lexicographic product of graphs,
 - series-parallel graphs.

Generalizations

In a hypergraph, an edge can join more than two vertices.

An undirected graph can be seen as a simplicial complex consisting of 1-simplices (the edges) and 0-simplices (the vertices). As such, complexes are generalizations of graphs since they allow for higher-dimensional simplices.

Every graph gives rise to a matroid.

In model theory, a graph is just a structure. But in that case, there is no limitation on the number of edges: it can be any cardinal number, see continuous graph.

In computational biology, power graph analysis introduces power graphs as an alternative representation of undirected graphs.

In geographic information systems, geometric networks are closely modeled after graphs, and borrow many concepts from graph theory to perform spatial analysis on road networks or utility grids.

See also

- Conceptual graph
- Dual graph
- Graph (abstract data type)
- Graph database
- Graph drawing
- List of graph theory topics
- List of publications in graph theory
- Network theory

Notes

1. Trudeau, Richard J. (1993). *Introduction to Graph Theory* (<http://store.doverpublications.com/0486678709.html>) (Corrected, enlarged republication. ed.). New York: Dover Pub. p. 19. ISBN 978-0-486-67870-2. Retrieved 8 August 2012. "A graph is an object consisting of two sets called its *vertex set* and its *edge set*."
2. See:
 - J. J. Sylvester (February 7, 1878) "Chemistry and algebra," (<https://books.google.com/books?id=KcoKAAAYAAJ&vq=Sylvester&pg=PA284#v=onepage&q&f=false>) *Nature*, 17 : 284. doi:10.1038/017284a0 (<https://doi.org/10.1038%2F017284a0>). From page 284: "Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph."
 - J. J. Sylvester (1878) "On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics, — with three appendices," (<https://books.google.com/books?id=1q0EAAAAYAAJ&pg=PA64#v=onepage&q&f=false>) *American Journal of Mathematics, Pure and Applied*, 1 (1) : 64–90 doi:10.2307/2369436 (<https://doi.org/10.2307%2F2369436>) .JSTOR 2369436 (<https://www.istor.org>)

(1) <https://doi.org/10.23694/2369436>. The term "graph" first appears in this paper on page 65.

3. Gross, Jonathan L.; Yellen, Jay (2004). *Handbook of graph theory* (https://books.google.com/?id=mKklGlea_BkC). CRC Press. p. 35 (https://books.google.com/books?id=mKklGlea_BkC&pg=PA35&lpg=PA35). ISBN 978-1-58488-090-5.
4. See, for instance, Iyanaga and Kawada, *69 J*, p. 234 or Biggs, p. 4.
5. See, for instance, Graham et al., p. 5.
6. For example, see. Bollobás, p. 7 and Diestel, p. 25.
7. Gross (1998), p. 3, Gross (2003), p. 205, Harary, p.10, and Zwillinger, p. 220.
8. For example, see Balakrishnan, p. 1, Gross (2003), p. 4, and Zwillinger, p. 220.
9. Fletcher, Peter; Hoyle, Hughes; Patty, C. Wayne (1991). *Foundations of Discrete Mathematics* (International student ed.). Boston: PWS-KENT Pub. Co. p. 463. ISBN 0-53492-373-9. "A weighted graph is a graph in which a number $w(e)$, called its *weight*, is assigned to each edge e ."
10. Strang, Gilbert (2005), *Linear Algebra and Its Applications* (<https://books.google.com/books?vid=ISBN0030105676>) (4th ed.), Brooks Cole, ISBN 0-03-010567-6
11. Lewis, John (2013), *Java Software Structures* (<https://www.amazon.com/Java-Software-Structures-Designing-Using/dp/0133250121>) (4th ed.), Pearson, p. 405, ISBN 0133250121
12. Grandjean, Martin (2016). "A social network analysis of Twitter: Mapping the digital humanities community" (<http://cogentoa.tandfonline.com/doi/full/10.1080/23311983.2016.1171458>). *Cogent Arts & Humanities*. 3 (1): 1171458. doi:10.1080/23311983.2016.1171458 (<https://doi.org/10.1080%2F23311983.2016.1171458>).
13. Pankaj Gupta, Ashish Goel, Jimmy Lin, Aneesh Sharma, Dong Wang, and Reza Bosagh Zadeh WTF: The who-to-follow system at Twitter (<http://dl.acm.org/citation.cfm?id=2488433>), *Proceedings of the 22nd international conference on World Wide Web*. doi:10.1145/2488388.2488433 (<https://doi.org/10.1145%2F2488388.2488433>).

References

- Balakrishnan, V. K. (1997-02-01). *Graph Theory* (1st ed.). McGraw-Hill. [ISBN 0-07-005489-4](#).
- Berge, Claude (1958). *Théorie des graphes et ses applications* (in French). Dunod, Paris: Collection Universitaire de Mathématiques, II. pp. viii+277. Translation: -. Dover, New York: Wiley. 2001 [1962].
- Biggs, Norman (1993). *Algebraic Graph Theory* (2nd ed.). Cambridge University Press. [ISBN 0-521-45897-8](#).
- Bollobás, Béla (2002-08-12). *Modern Graph Theory* (1st ed.). Springer. [ISBN 0-387-98488-7](#).
- Bang-Jensen, J.; Gutin, G. (2000). *Digraphs: Theory, Algorithms and Applications* (<http://www.cs.rhul.ac.uk/books/dbook/>). Springer.
- Diestel, Reinhard (2005). *Graph Theory* (<http://diestel-graph-theory.com/GrTh.html>) (3rd ed.). Berlin, New York: Springer-Verlag. [ISBN 978-3-540-26183-4](#).
- Graham, R.L.; Grötschel, M.; Lovász, L, eds. (1995). *Handbook of Combinatorics*. MIT Press. [ISBN 0-262-07169-X](#).
- Gross, Jonathan L.; Yellen, Jay (1998-12-30). *Graph Theory and Its Applications*. CRC Press. [ISBN 0-8493-3982-0](#).
- Gross, Jonathan L.; Yellen, Jay, eds. (2003-12-29). *Handbook of Graph Theory*. CRC. [ISBN 1-58488-090-2](#).
- Harary, Frank (January 1995). *Graph Theory*. Addison Wesley Publishing Company. [ISBN 0-201-41033-8](#).
- Iyanaga, Shôkichi; Kawada, Yukiyo (1977). *Encyclopedic Dictionary of Mathematics*. MIT Press. [ISBN 0-262-09016-3](#).
- Zwillinger, Daniel (2002-11-27). *CRC Standard Mathematical Tables and Formulae* (31st ed.). Chapman &

Further reading

- Trudeau, Richard J. (1993). *Introduction to Graph Theory* (<http://store.doverpublications.com/0486678709.html>) (Corrected, enlarged republication. ed.). New York: Dover Publications. ISBN 978-0-486-67870-2. Retrieved 8 August 2012.

External links

- Weisstein, Eric W. "Graph" (<http://mathworld.wolfram.com/Graph.html>). *MathWorld*.
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